Introduction

In what follows are three different proofs of various forms of the Generalised Binomial Theorem. The first two focus on negative coefficients, and the third works for all $n \in \mathbb{R}$.

We first introduce generalised Binomial coefficients, which since the numerator is a non-negative integer work for all values of n

Definition (Generalised Binomial Coefficients)**.**

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}, n \in \mathbb{C}, k \in \mathbb{Z}_{\geq 0}$$

The Generalised Binomial Theorem states that:

Theorem (Generalised Binomial Theorem)

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k, |x| < 1$$

ie in the formula book:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}x^r + \cdots, |x| < 1$$

Negative Binomial Theorem - No Calculus

This proof relies on no calculus at all, but does require us to prove a useful combinatorial result.

Lemma (Hockey Stick Identity)

$$\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\cdots+\binom{n+r}{r}=\binom{n+r+1}{r}$$

Proof.

$$\begin{aligned}
LHS &= \boxed{\binom{n}{0}}+\binom{n+1}{1}+\binom{n+2}{2}+\cdots+\binom{n+r}{r} \\
&= \binom{\mathbf{n+1}}{\mathbf{0}}+\binom{n+1}{1}+\binom{n+2}{2}+\cdots+\binom{n+r}{r} && \left(\text{since } \tbinom{n}{0}=1=\tbinom{n+1}{0}\right) \\
&= \boxed{\binom{n+1}{0}+\binom{n+1}{1}}+\binom{n+2}{2}+\cdots+\binom{n+r}{r} \\
&= \binom{\mathbf{n+2}}{\mathbf{1}}+\binom{n+2}{2}+\cdots+\binom{n+r}{r} && \left(\text{since } \tbinom{m}{i}+\tbinom{m}{i+1}=\tbinom{m+1}{i+1}\right) \\
&= \boxed{\binom{n+2}{1}+\binom{n+2}{2}}+\cdots+\binom{n+r}{r} \\
&= \binom{\mathbf{n+3}}{\mathbf{2}}+\cdots+\binom{n+r}{r} && \left(\text{since } \tbinom{m}{i}+\tbinom{m}{i+1}=\tbinom{m+1}{i+1}\right) \\
&= \boxed{\binom{n+r}{r-1}+\binom{n+r}{r}} && (\text{repeatedly doing this trick}) \\
&= \binom{n+r+1}{r} && \left(\text{since } \tbinom{m}{i}+\tbinom{m}{i+1}=\tbinom{m+1}{i+1}\right) \\
&= RHS
\end{aligned}$$

□

Looking at Pascal's triangle we can see (perhaps) why this is called the Hockey stick identity. We want to sum a strip like the blue boxes, and it should be equal to the entry where the red box would be.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

The first 1 and 2 combine to become 3. Then this 3 and 4 combine to become the 6, Which finally combines with 4 to become the 10

Theorem (Binomial Series Expansion - Version 1)

$$\begin{aligned}(1-x)^{-n} &= 1+\binom{n}{1}x+\binom{n+1}{2}x^2+\binom{n+2}{3}x^3+\cdots+\binom{n+i-1}{i}x^i+\cdots \\ &= \sum_{i=0}^{\infty}\binom{n-1+i}{i}x^i\end{aligned}$$

Proof. We proceed by induction. Base case $n=1$.

The coefficients are $\binom{n-1+i}{i}=\binom{i}{i}=1$, so this is the well-known series for a geometric progression.

Inductive step. Assume our statement is true for $n=k$, and consider for $n=k+1$.

So

$$(1-x)^{-k} = 1+\binom{k}{1}x+\binom{k+1}{2}x^2+\binom{k+2}{3}x^3+\cdots+\binom{k+i-1}{i}x^i+\cdots$$

$$\begin{aligned}(1-x)^{-(k+1)} &= (1-x)^{-1}(1-x)^{-k} \\ &= (1+x+x^2+\cdots)\left(1+\binom{k}{1}x+\binom{k+1}{2}x^2+\binom{k+2}{3}x^3+\cdots+\binom{k-1+i}{i}x^i+\cdots\right) \\ &= 1+\left(\binom{k}{1}+\binom{k-1}{0}\right)x+\left(\binom{k+1}{2}+\binom{k}{1}+\binom{k-1}{0}\right)x^2+\cdots \\ &\quad +\cdots+\left(\binom{k-1+i}{i}+\cdots+\binom{k+1}{2}+\binom{k}{1}+\binom{k-1}{0}\right)x^i+\cdots \\ &\qquad \text{(by multiplying out the infinite series, consider all the ways we can obtain } x^j\text{)} \\ &= 1+\binom{k+1}{1}x+\binom{k+2}{2}x^2+\cdots+\binom{k-1+i+1}{i}x^i+\cdots \\ &\qquad \text{(using the Hockey stick identity at every point, with } n=k-1, r=i+1\text{)} \\ &= 1+\binom{(k+1)+0}{1}x+\binom{(k+1)+1}{2}x^2+\cdots+\binom{(k+1)+i-1}{i}x^i+\cdots\end{aligned}$$

Which is exactly what we wanted to prove for $n=k+1$.

Therefore since our hypothesis is true for $n=1$, and if it is true for $n=k$ it is true for $n=k+1$, by the Principle of Mathematical Induction, it is true for all $n \geq 1$ □

Lemma (Negative Binomial Coefficients)

$$\binom{-n}{r} = (-1)^r \binom{n+r-1}{r}$$

Proof.

$$\begin{aligned}\binom{-n}{r} &= \frac{(-n)(-n-1)(-n-2)\cdots(-n-(r-1))}{r!} \\ &= \frac{(-1)^r n(n+1)(n+2)\cdots(n+r-1)}{r!} \\ &= \frac{(-1)^r (n+r-1)!}{(n-1)!r!} \\ &= (-1)^r \binom{n+r-1}{r}\end{aligned}$$

□

This is useful, because we can now re-write our theorem as a new version:

Theorem (Binomial Series Expansion - Version 2)

$$\begin{aligned}(1+x)^{-n} &= 1 + \binom{-n}{1}x + \binom{-n}{2}x^2 + \binom{-n}{3}x^3 + \cdots + \binom{-n}{i}x^i + \cdots \\ &= \sum_{i=0}^{\infty} \binom{-n}{i} x^i\end{aligned}$$

Proof.

$$\begin{aligned}(1+x)^{-n} &= (1-(-x))^{-n} \\ &= 1 + \binom{n}{1}(-x) + \binom{n+1}{2}(-x)^2 + \binom{n+2}{3}(-x)^3 + \cdots + \binom{n+i-1}{i}(-x)^i + \cdots && \text{(by the first version)} \\ &= 1 + (-1)\binom{-n}{1}(-x) + (-1)^2\binom{-n}{2}(-x)^2 + (-1)^3\binom{-n}{3}(-x)^3 + \cdots \\ &\qquad \cdots + (-1)^i\binom{-n}{i}(-x)^i + \cdots && \text{(By our lemma about negative binomial coefficients)} \\ &= 1 + \binom{-n}{1}x + \binom{-n}{2}x^2 + \binom{-n}{3}x^3 + \cdots + (-1)^{2i}\binom{-n}{i}x^i + \cdots \\ &= 1 + \binom{-n}{1}x + \binom{-n}{2}x^2 + \binom{-n}{3}x^3 + \cdots + \binom{-n}{i}x^i + \cdots && \text{(since all } -1\text{s are being raised to even powers)}\end{aligned}$$

□

Negative Binomial Theorem - Mark II

This proof is much simpler, since it doesn't rely on the hockey stick identity, which can be tricky to grasp if you've not seen it before. However, the first step in the proof involves the chain rule, which some students haven't necessarily encountered yet.

Lemma (Product Formula)

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

Proof.

$$\begin{aligned} LHS &= \binom{n}{k} \\ &= \frac{n!}{k!(n-k)!} \\ &= \frac{n(n-1)!}{k(k-1)!(n-k)!} \\ &= \frac{n}{k}\frac{(n-1)!}{(k-1)!(n-1-(k-1))!} \\ &= \frac{n}{k}\binom{n-1}{k-1} \\ &= RHS \end{aligned}$$

□

This can also be written as $k\binom{n}{k} = n\binom{n-1}{k-1}$

Lemma (Useful Binomial Coefficient recurrence)

$$\frac{n}{n-k}\binom{n-1}{k} = \binom{n}{k}$$

Proof.

$$\begin{aligned} LHS &= \frac{n}{n-k}\binom{n-1}{k} \\ &= \frac{n}{n-k}\frac{(n-1)!}{k!(n-1-k)!} \\ &= \frac{n(n-1)!}{k!(n-k)(n-k-1)!} \\ &= \frac{n!}{k!(n-k)!} \\ &= \binom{n}{k} \\ &= RHS \end{aligned}$$

□

Theorem (Binomial Series Expansion - Version 1)

$$\begin{aligned}(1-x)^{-n} &= 1+\binom{n}{1}x+\binom{n+1}{2}x^2+\binom{n+2}{3}x^3+\cdots+\binom{n+i-1}{i}x^i+\cdots\\ &= \sum_{i=0}^{\infty}\binom{n-1+i}{i}x^i\end{aligned}$$

Proof. We proceed by induction. Base case $n=1$.

$$(1-x)^{-1} = 1+x+\cdots+x^i+\cdots$$

by the formula for the geometric progression.

Inductive step. Assume for $n=k$, ie

$$(1-x)^{-k} = 1+\binom{k}{1}x+\binom{k+1}{2}x^2+\binom{k+2}{3}x^3+\cdots+\binom{k+i-1}{i}x^i+\cdots$$

Differentiating both sides with respect to x, we get

$$\begin{aligned}(-k)(1-x)^{-k-1}\cdot(-1) &= k+2\binom{k+1}{2}x+3\binom{k+2}{3}x^2+\cdots+i\binom{k+i-1}{i}x^{i-1}+\cdots && \text{(differentiating termwise)}\\ k(1-x)^{-k-1} &= k+(k+1)\binom{k}{1}x+(k+2)\binom{k+1}{2}x^2+\cdots\\ &\quad\cdots+(k+i-1)\binom{k+i-2}{i-1}x^{i-1}+\cdots && \text{(applying } k\tbinom{n}{k}=n\tbinom{n-1}{k-1}\text{)}\\ (1-x)^{-k-1} &= 1+\frac{k+1}{k}\binom{k}{1}x+\frac{k+2}{k}\binom{k+1}{2}x^2+\cdots\\ &\quad\cdots+\frac{k+i-1}{k}\binom{k+i-2}{i-1}x^{i-1}+\cdots && \text{(dividing by } k\text{)}\\ (1-x)^{-(k+1)} &= 1+\binom{k+1}{1}x+\binom{k+2}{2}x^2+\cdots+\binom{k+i-1}{i-1}x^{i-1}+\cdots && \text{(applying } \tfrac{n}{n-k}\tbinom{n-1}{k}=\tbinom{n}{k}\text{)}\end{aligned}$$

But this is exactly our statement when $n=k+1$.

Therefore, since our statement is true when $n=1$ and if $n=k$ is true, $n=k+1$ is true, our statement is true for all $n\geq 1$ □

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$$\begin{aligned}\binom{-n}{r} &= \frac{(-n)(-n-1)(-n-2)\cdots(-n-(r-1))}{r!} \\ &= \frac{(-1)^r n(n+1)(n+2)\cdots(n+r-1)}{r!} \\ &= \frac{(-1)^r (n+r-1)!}{(n-1)!r!} \\ &= (-1)^r \binom{n+r-1}{r}\end{aligned}$$

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This is useful, because we can now re-write our theorem as a new version:

Theorem (Binomial Series Expansion - Version 2)

$$\begin{aligned}(1+x)^{-n} &= 1 + \binom{-n}{1}x + \binom{-n}{2}x^2 + \binom{-n}{3}x^3 + \cdots + \binom{-n}{i}x^i + \cdots \\ &= \sum_{i=0}^{\infty} \binom{-n}{i} x^i\end{aligned}$$

Proof.

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□

Rational Binomial Coefficients

Motivated by our success looking at negative binomial coefficients, we might hypothesis that $(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$.

We can prove this for rationals (and using a fairly weak argument for all reals) using a relatively straightforward method.

First, let $f(\alpha) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$, we should be comfortable that this is a reasonably well behaved for small x by comparison with geometric series.

Lemma (f satisfies Cauchy's functional equation)

$$f(\alpha)f(\beta) = f(\alpha+\beta)$$

Proof.

$$\begin{aligned} f(\alpha)f(\beta) &= \left(\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n\right)\left(\sum_{n=0}^{\infty} \binom{\beta}{n} x^n\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{\alpha}{k}\binom{\beta}{n-k}\right) x^n \end{aligned}$$

So it suffices to prove that

Lemma (Chu-Vandemonde)

$$\sum_{k=0}^{n} \binom{\alpha}{k}\binom{\beta}{n-k} = \binom{\alpha+\beta}{n}$$

Proof. By induction. We proceed by induction on n. When $n = 0$ it is clearly true.

Assume the formula is true for some $n = l$, then consider $n = l+1$.

$$\begin{aligned} \binom{\alpha+\beta}{l+1} &= \frac{(\alpha+\beta-l)}{l+1}\binom{\alpha+\beta}{l} \\ &= \frac{(\alpha+\beta-l)}{l+1}\sum_{k=0}^{l}\binom{\alpha}{k}\binom{\beta}{l-k} && \text{(inductive hypothesis)} \\ &= \sum_{k=0}^{l}\frac{(\alpha-k)+(\beta-(l-k))}{l+1}\binom{\alpha}{k}\binom{\beta}{l-k} \\ &= \sum_{k=0}^{l}\left(\frac{k+1}{l+1}\binom{\alpha}{k+1}\binom{\beta}{l-k} + \frac{l-k+1}{l+1}\binom{\alpha}{k}\binom{\beta}{l-k+1}\right) && \left(\binom{n}{k} = \tfrac{n}{k}\binom{n-1}{k-1}\right) \\ &= \binom{\alpha}{l+1} + \binom{\beta}{l+1} + \sum_{k=1}^{l}\left(\frac{k}{l+1} + \frac{l-k+1}{l+1}\right)\binom{\alpha}{k}\binom{\beta}{l-k+1} && \text{(reindexing the sum)} \\ &= \binom{\alpha}{l+1} + \binom{\beta}{l+1} + \sum_{k=1}^{l}\binom{\alpha}{k}\binom{\beta}{l-k+1} \\ &= \sum_{k=0}^{l+1}\binom{\alpha}{k}\binom{\beta}{l-k+1} \end{aligned}$$

Which is exactly our statement when $n = l + 1$.

Therefore, since our statement is true when $n = 0$ and if $n = l$ is true, $n = l + 1$ is true, our statement is true for all $n \geq 0$

$\square$

So returning to our algebra from earlier:

$$\begin{aligned} f(\alpha)f(\beta) &= \left(\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n\right)\left(\sum_{n=0}^{\infty}\binom{\beta}{n}x^n\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{\alpha}{k}\binom{\beta}{n-k}\right)x^n \\ &= \sum_{n=0}^{\infty}\binom{\alpha+\beta}{n}x^n \\ &= f(\alpha+\beta) \end{aligned}$$

$\square$

Since we have shown that $f(\alpha)f(\beta) = f(\alpha+\beta)$ for any value of α, β, lets consider some interesting ones. For example, $f\left(\frac{1}{2}\right)f\left(\frac{1}{2}\right) = f(1) = (1+x)$. Therefore it's straightforward to see that $f\left(\frac{1}{2}\right) = (1+x)^{\frac{1}{2}}$.

Indeed, since we know by the binomial theorem, $f(n) = (1+x)^n$, we know that $f\left(\frac{p}{q}\right)^q = f(p) = (1+x)^p \Rightarrow f(\frac{p}{q}) = (1+x)^{\frac{p}{q}}$.

Since we also know that $f(-1) = (1+x)^{-1}$ (by consideration of geometric series), we can also discover that $(1+x)^{-n} = f(-1)^n = f(-n)$, so we have also proved our result for negative binomial coefficients. And by the same logic $f(-\frac{p}{q})^q = f(-p) = (1+x)^{-p}$ therefore $f(-\frac{p}{q}) = (1+x)^{-\frac{p}{q}}$. Therefore we have proven the Generalised Binomial Theorem for all $n \in \mathbb{Q}$

It doesn't take too much work from here to turn this into a proof for all $n \in \mathbb{R}$, it would suffice to show that $f(\alpha)$ is continuous.